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# PROCEEDINGS

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## HORIZONTALLY CURVED BOX BEAMS

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STRUCTURAL DIVISION

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PAPERS

# HORIZONTALLY CURVED BOX BEAMS

By Charles E. Cutts, A. M. ASCE

## Synopsis

A beam curved horizontally through an angle of 90° and resting on four supports is analyzed for maximum torque, shear, moment, deflection, and rotation when the beam carries a uniform load. Three types of curved box beams were fabricated and tested with five concentrated loads placed symmetrically on the curved part of the beam to assimilate uniform loading. Unit strains were measured at critical points on the beam by means of SR-4 electrical strain gages.

A new method of design for closed section curved beams and a theoretical development for determining rotation and longitudinal warping stresses are included herein.

## Introduction

The shape of the cross section of rolled steel sections, such as the I-sections and channel sections, have been designed for maximum resistance of the section to bending moment. These sections do not resist torsion well and do not utilize the material economically for this purpose. Inge Lyse and Bruce G. Johnston (11*a*)[2] have indicated the relative torsional rigidity and torsional strength of various sections (see Fig. 1.)

Since closed steel sections have such large values of torsional rigidity and strength, appreciable reduction in shearing stresses and rotation due to torsion can be obtained by building up a closed section member. Fig. 2 indicates the three types of closed sections tested—(*a*) consisting of two channels; (*b*) consisting of four plates; and (*c*) consisting of an I-beam with side plates. The two-channel section (Fig. 2(*a*)) was joined by intermittent butt welds and the other two sections by intermittent fillet welds. The welding consisted of 7-in. welds with 3 in. clear distances between them.

Fig. 3 illustrates one of the uses of the horizontally curved beam. In this building the corner column has been eliminated and the horizontally curved

Note.—Written comments are invited for publication; the last discussion should be submitted by November 1, 1952.

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[2] Numerals in parentheses, thus: (11*a*), refer to corresponding numbers in the bibliography, Appendix I.

beam spans the gap. The curved corner in this illustration is believed to be supported by cantilever construction as noted at the top of the building. This corner could be designed by using a closed-section beam continuous past the first columns from the corner and held down at the second columns, eliminating the supporting cantilever construction.

| SECTIONS | RIGIDITY | STRENGTH |
|---|---|---|
| (a) ● | 100.0 | 100.0 |
| (b) ○ | 637.0 | 332.0 |
| (c) C— | 5.5 | 18.0 |
| (d) ▲ | 70.0 | 62.0 |
| (e) ■ | 88.0 | 74.0 |
| (f) □ | 341.0 (Approx.) | 280.0 (Approx.) |
| (g) I | 9.9 (Nearly Exact) | 22.2 (Approx.) |
| (h) I $r=w=\frac{2}{3}n$ | 11.6 (Nearly Exact) | 22.8 (Approx.) |
| (i) I Fixed Ends 6.5 In. 1.5 Ft | 78.1 (Approx.) | 38.3 (Approx.) |

FIG. 1

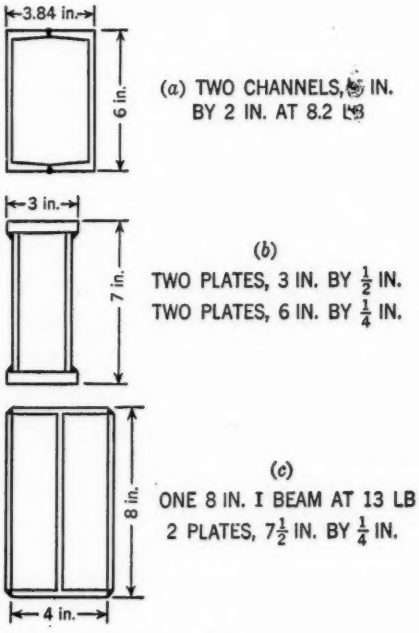
3.84 in.
6 in.
(a) TWO CHANNELS, 6 IN. BY 2 IN. AT 8.2 LB
3 in.
7 in.
(b)
TWO PLATES, 3 IN. BY ½ IN.
TWO PLATES, 6 IN. BY ¼ IN.
8 in.
(c)
ONE 8 IN. I BEAM AT 13 LB
2 PLATES, 7½ IN. BY ¼ IN.
4 in.

FIG. 2

## THEORETICAL ANALYSIS

*Notation.*—The letter symbols adopted for use in this paper are defined where they first appear, in the illustrations or in the text, and are arranged alphabetically for reference in Appendix II.

The beam is continuous, and is supported vertically at points A, B, C, D as shown in Fig. 4. These four supports remain at the same elevation and offer no resistance to flexural or torsional rotation. The section is assumed to be compact, and secondary effects are omitted from the general equilibrium equations. The secondary effects are considered separately in the second part of this analysis.

The uniform load of $w$ lb per ft is distributed from point A to point B on the curved part of the beam. Because of the symmetry of the beam and loading, by the law of statics $\Sigma V = 0$, and shears at reactions $A$ and $B$ are equal to one half of the load on the beam:

$$V_A = V_B = \frac{w \pi r}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

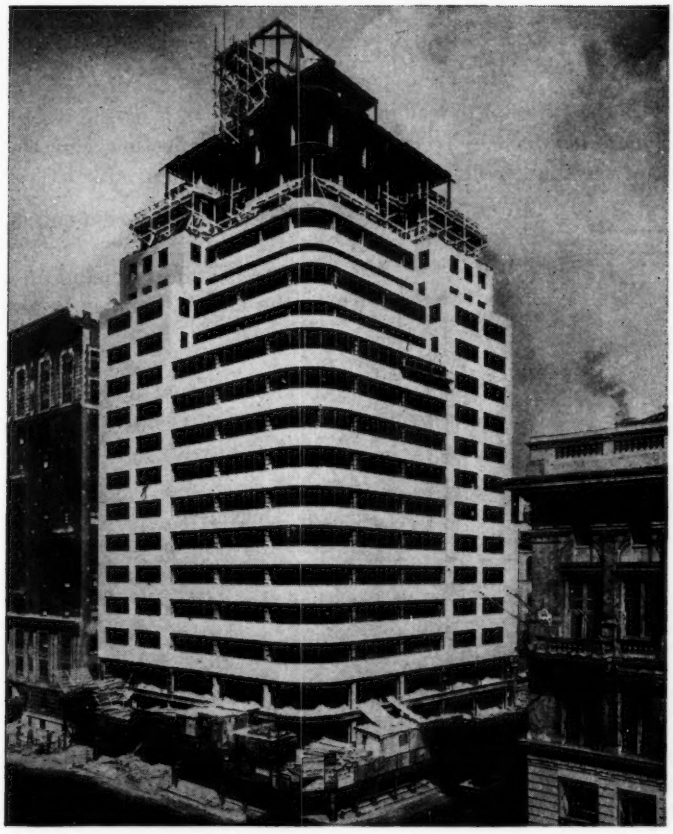

FIG. 3

Taking $\Sigma M = 0$ about the axis BD:

$$M_A = -\frac{\pi r^2}{4} w + \int_0^{\pi/2} w r^2 (1 - \cos \alpha)\, d\alpha = -\left(1 - \frac{\pi}{4}\right) w r^2 \quad (2a)$$

$$M_A = M_B = -0.2146\, w r^2 \quad (2b)$$

$$M_\theta = -\left(1 - \frac{\pi}{4}\right) w r^2 \cos \theta + \frac{w \pi r}{4} \times r \sin \theta - \int_0^\theta w r^2 \sin \alpha\, d\alpha \quad (2c)$$

$$M_\theta = w r^2 \left(\frac{\pi}{4} \sin \theta + \frac{\pi}{4} \cos \theta - 1\right) \quad (2d)$$

$$M_E = +\, w r^2 \left(\frac{\sqrt{2}\,\pi}{4} - 1\right) = +\, 0.1107\, w r^2 \quad (2e)$$

$$T_\theta = M_B \sin \theta + \frac{w \pi r^2}{4} (1 - \cos \theta) - \int_0^\theta w r^2\, d\alpha\, (1 - \cos \alpha) \quad (2f)$$

$$T_\theta = w r^2 \left[\frac{\pi}{4} (1 + \sin \theta - \cos \theta) - \theta\right] \quad (2g)$$

$$T = 0 \text{ (at points A, B, and E)} \quad (2h)$$

and

$$T_{max} = 0.03312\, w\, r^2 \text{ (at } \theta = 19° - 12')\ldots\ldots\ldots

to $\int \frac{T\,ds}{G\,J}$ multiplied by the torque arm $r\left[1 - \cos\left(\frac{\pi}{4} - \alpha\right)\right]$; thus,

$$T = w\,r^2\left[\frac{\pi}{4}(1 + \sin\alpha - \cos\alpha) - \alpha\right] \quad \ldots\ldots(4)$$

and

$$\delta\ (\text{center}) = \int_0^{\pi/4} \frac{T\,ds}{G\,J}\,r\left[1 - \cos\left(\frac{\pi}{4} - \alpha\right)\right] \quad \ldots\ldots(5a)$$

$$\delta\ (\text{center}) = \frac{w\,r^4}{G\,J}\int_0^{\pi/4}\left[\frac{\pi}{4}(1 + \sin\alpha - \cos\alpha)\right]\left[1 - \cos\left(\frac{\pi}{4} - \alpha\right)\right] \quad \ldots\ldots(5b)$$

and

$$\delta\ (\text{center}) = -\,0.0016845\,\frac{w\,r^4}{G\,J}\ (\text{upward}) \quad \ldots\ldots(5c)$$

in which (10)

$$J = \frac{4\,A_s^2}{\oint \frac{ds}{t}} \quad \ldots\ldots(6)$$

and $ds$ is a short increment of length and $A_s$ is the shear area.

*Rotation and Stresses Resulting from Warping of the Section Due to Torsion.*— In this development, the beam is analyzed for one half of its curved length since symmetrical stress conditions exist in the two 45° halves of the curved part. This solution is approximate in that an equivalent length of straight beam is analyzed and that the variation in torque is assumed to be a function of the sine curve.

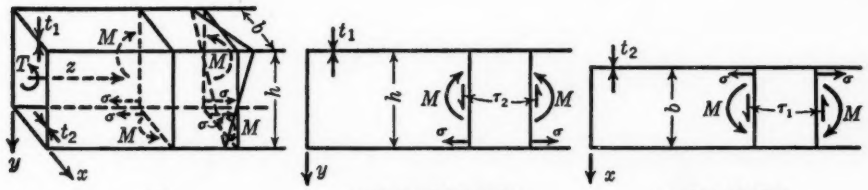


FIG. 5

When torque is applied to a section as shown in Fig. 5, in addition to shearing stresses, $\tau$, resisting the torque there appear longitudinal stresses as well. These longitudinal stresses are caused by the warping of the section and are related in part to the wall thicknesses and side dimensions. By the double integration method (16),

$$\frac{d^2x}{dz^2} = \frac{M_y}{E\,I_y} - \frac{1}{G}\frac{d\tau_1}{dz} \quad \ldots\ldots(7a)$$

and

$$\frac{d^2y}{dz^2} = -\frac{M_x}{E\,I_x} - \frac{1}{G}\frac{d\tau_2}{dz} \quad \ldots\ldots (7b)$$

Since the unit bending stress $\sigma = \frac{M_c}{I}$,

$$\frac{d^2x}{dz^2} = \frac{2\,\sigma}{b\,E} - \frac{1}{G}\frac{d\tau_1}{dz} \quad \ldots\ldots (8a)$$

$$\frac{d^2y}{dz^2} = -\frac{2\,\sigma}{h\,E} - \frac{1}{G}\frac{d\tau_2}{dz} \quad \ldots\ldots (8b)$$

$$q = \tau_1 t_1 = \tau_2 t_2 \quad \ldots\ldots (8c)$$

$$t_1\,\tau_1 = \frac{d}{dz}\left(\frac{2\,\sigma}{b}\,S_1\right) \quad \ldots\ldots (8d)$$

and

$$t_2\,\tau_2 = -\frac{d}{dz}\left(\frac{2\,\sigma}{h}\,S_2\right) \quad \ldots\ldots (8e)$$

in which $q$ denotes shear flow and $S$ is the section modulus. The maximum values of shear flow become

$$t_1\,\tau_1 = \frac{1}{4}\frac{d}{dz}(\sigma\,t_1\,b) + q \quad \ldots\ldots (9a)$$

and

$$t_2\,\tau_2 = -\frac{1}{4}\frac{d}{dz}(\sigma\,t_2\,h) + q \quad \ldots\ldots (9b)$$

Since (with $\phi$ = angle of rotation) $x = \phi\frac{h}{2}$ and $y = \phi\frac{b}{2}$,

$$\frac{1}{2}\frac{d^2(\phi\,h)}{dz^2} = \frac{2\,\sigma}{b\,E} - \frac{1}{4\,G}\frac{d}{dz}\left[\frac{1}{t_1}\frac{d}{dz}(\sigma\,t_1\,b)\right] - \frac{1}{G}\frac{d}{dz}\left(\frac{q}{t_1}\right) \quad \ldots\ldots (10a)$$

and

$$\frac{1}{2}\frac{d^2(\phi\,b)}{dz^2} = -\frac{2\,\sigma}{h\,E} + \frac{1}{4\,G}\frac{d}{dz}\left[\frac{1}{t_2}\frac{d}{dz}(\sigma\,t_2\,h)\right] - \frac{1}{G}\frac{d}{dz}\left(\frac{q}{t_2}\right) \quad \ldots\ldots (10b)$$

and equating these expressions,

$$\frac{4\,\sigma}{E\,h\,b} - \frac{\sigma''\,b}{2\,G\,h} - \frac{2\,q'}{G\,t_1\,h} = -\frac{4\,\sigma}{E\,h\,b} + \frac{\sigma''\,h}{2\,G\,b} - \frac{2\,q'}{G\,t_2\,b} \quad \ldots\ldots (11a)$$

$$\frac{\sigma''}{2\,G}\left(\frac{h}{b} + \frac{b}{h}\right) - \frac{8\,\sigma}{E\,h\,b} - \frac{2\,q'}{G}\left(\frac{1}{t_2\,b} - \frac{1}{t_1\,h}\right) = 0 \quad \ldots\ldots (11b)$$

$$T = 2\left(\int_{-b/2}^{b/2}\tau_1\,t_1\,\frac{h}{2}\,dx + \int_{-h/2}^{h/2}\tau_2\,t_2\,\frac{b}{2}\,dy\right) \quad \ldots\ldots (11c)$$

and

$$T = 2\,q\,b\,h + \frac{\sigma'\,b\,h}{6}\,(b\,t_1 - h\,t_2) \quad \ldots\ldots (11d)$$

Solving for $q$,

$$q = \frac{T}{2\,b\,h} - \frac{\sigma'}{12}\,(b\,t_1 - h\,t_2) \quad \ldots\ldots (12)$$

Since the torque varies in a manner similar to the function $\sin \frac{\pi z}{l}$,

$$T = T_{\max} \sin \frac{\pi z}{l} \quad \ldots\ldots (13)$$

in which $l$ is one half the span length and the following differential equation is obtained:

$$\sigma'' - a^2\sigma + C \cos \frac{\pi z}{l} = 0 \quad \ldots\ldots (14)$$

in which

$$a^2 = \frac{\dfrac{48\,G}{E}}{b\,h\left[3\left(\dfrac{h}{b} + \dfrac{b}{h}\right) + \left(\dfrac{1}{t_2\,b} - \dfrac{1}{t_1\,h}\right)\left(b\,t_1 - h\,t_2\right)\right]} \quad \ldots\ldots (15a)$$

and

$$C = \frac{-\dfrac{6\,\pi\,T_{\max}}{l}\left(\dfrac{1}{t_2\,b} - \dfrac{1}{t_1\,h}\right)}{b\,h\left[3\left(\dfrac{h}{b} + \dfrac{b}{h}\right) + \left(\dfrac{1}{t_2\,b} - \dfrac{1}{t_1\,h}\right)\left(b\,t_1 - h\,t_2\right)\right]} \quad \ldots\ldots (15b)$$

The general solution to Eq. 14 becomes

$$\sigma = \frac{C}{\dfrac{\pi^2}{l^2} + a^2} \cos \frac{\pi z}{l} + \mathrm{A} \sinh a\,z + B \cosh a\,z \quad \ldots\ldots (16)$$

If $\sigma = 0$ at the center of the span where, $z = 0$,

$$B = -\frac{C}{\dfrac{\pi^2}{l^2} + a^2} \quad \ldots\ldots (17)$$

and, since $\sigma' = 0$ at $z = l$,

$$A = \frac{C}{\dfrac{\pi^2}{l^2} + a^2}\left(\frac{e^{al} + e^{-al}}{e^{al} - e^{-al}}\right) \quad \ldots\ldots (18)$$

By substituting Eq. 16 in

$$\phi'' = \frac{4\,\sigma}{b\,h\,E} - \frac{b}{2\,G\,h}\,\sigma'' - \frac{2}{G\,t_1\,h}\,q' \quad \ldots\ldots (19)$$

and integrating,

$$\phi' = \frac{4}{E\,h\,b}\left(\frac{\frac{l}{\pi}C}{\frac{\pi^2}{l^2}+a^2}\sin\frac{\pi z}{l} + \frac{A}{a}\cosh a z + \frac{B}{a}\sinh a z\right)$$

$$- \frac{b}{2\,G\,h}\left(\frac{-\frac{\pi}{l}C}{\frac{\pi^2}{l^2}+a^2}\sin\frac{\pi z}{l} + A\,a\cosh a z + B\,a\sinh a z\right)$$

$$- \frac{2}{G\,t_1\,h}q + D \quad \ldots(20)$$

Since $\phi' = 0$ at $z = 0$,

$$D = -\frac{4\,A}{E\,h\,b\,a} + \frac{a\,b\,A}{2\,G\,h} \quad \ldots(21)$$

Integrating again,

$$\phi = \frac{4}{E\,h\,b}\left(-\frac{\frac{l^2}{\pi^2}C}{\frac{\pi^2}{l^2}+a^2}\cos\frac{\pi z}{l} + \frac{A}{a^2}\sinh a z + \frac{B}{a^2}\cosh a z\right)$$

$$- \frac{b}{2\,G\,h}\left(\frac{C}{\frac{\pi^2}{l^2}+a^2}\cos\frac{\pi z}{l} + A\sinh a z + B\cosh a z\right)$$

$$- \frac{2}{G\,t_1\,h}\left[-\frac{T_{\max}\frac{l}{\pi}\cos\frac{\pi z}{l}}{2\,b\,h} - \frac{\sigma}{12}(b\,t_1 - h\,t_2)\right] + D\,z + E \quad \ldots(22)$$

when $z = l$ and $\phi = 0$:

$$E = -\frac{4}{E\,h\,b}\left[\frac{\frac{l^2}{\pi^2}C}{\frac{\pi^2}{l^2}+a^2} + \frac{A}{a^2}\left(\frac{e^{al}-e^{-al}}{2}\right) + \frac{B}{a^2}\left(\frac{e^{al}-e^{-al}}{2}\right)\right]$$

$$+ \frac{b}{2\,G\,h}\left[-\frac{C}{\frac{\pi^2}{l^2}+a^2} + A\left(\frac{e^{al}-e^{-al}}{2}\right) + B\left(\frac{e^{al}-e^{-al}}{2}\right)\right]$$

$$+ \frac{2}{G\,t_1\,h}\left\{\frac{T_{\max}\frac{l}{\pi}}{2\,b\,h} - \left(\frac{b\,t_1 - h\,t_2}{12}\right)\left[-\frac{C}{\frac{\pi^2}{l^2}+a^2} + A\left(\frac{e^{al}-e^{-al}}{2}\right)\right.\right.$$

$$\left.\left. + B\left(\frac{e^{al}+e^{-al}}{2}\right)\right]\right\} - D\,l \quad \ldots(23)$$

and

$$\phi \text{ (center)} = \frac{4}{E\,h\,b}\left(-\frac{\frac{l^2}{\pi^2}C}{\frac{\pi^2}{l^2}+a^2}+\frac{B}{a^2}\right)-\frac{b}{2\,G\,h}\left(\frac{C}{\frac{\pi^2}{l^2}+a^2}+B\right)$$
$$-\frac{2}{G\,t_1\,h}\left(-\frac{T_{\max}\,l/\pi}{2\,b\,h}\right)+E\,..(24)$$

The hyperbolic functions cancel out for the ordinary span lengths used in building construction, thus greatly simplifying these expressions. A numerical example illustrating the application of the foregoing theory is presented in Appendix III.

## Experimental Procedure and Results

Fig. 6 indicates the plan and side views of the typical specimen tested in this experiments. Section A-A, Fig. 6(*a*) refers to either of the three sections in Fig. 2. The cost of the members varied from \$126 to \$170. Welding the

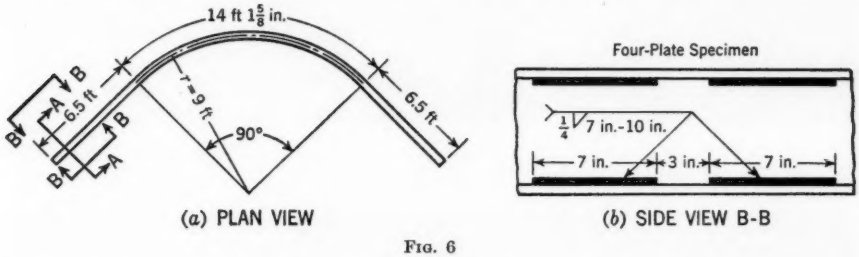


Fig. 6

members together provided considerable saving in cost compared to that of members joined by riveting.

Fig. 7 shows the layout plan for testing in which the box beams were supported by 3-in. square bearing plates at the intermediate supports and held down at the ends by loaded platforms. The curved beam was divided into six equal parts, and five equal concentrated loads were applied at 15° intervals. The loading was applied by means of lever arms with a platform carrying sacks of shot suspended at the outer end of each lever arm. Rotation at the center of the beam was measured by means of an optical lever. Vertical deflection of the beam was measured by Ames dials. A-1 and AR-1 (rosette type) electrical strain gages were located at twenty-two points to measure the unit strains. Fig. 8 shows the wiring arrangement with leads attached to a switching control near the K-box.

The total loading on the beams varied as indicated in line 2, Table 1. These loadings were calculated to produce a unit bending stress over the supports of 20,000 lb per sq in.

It is interesting to compare the unit stresses in flexure and shear with those calculated by the theory (lines 3, 4, and 5, Table 1).

The measured deflections of the beams varied from 0.40 in. to 0.55 in., and measured rotations at the center of the beam varied from 1.1° to 1.3°. The

computed deflections varied from 0.33 in. to 0.48 in. and the computed rotations at the center of beam varied from 0.64° to 0.99°.

The measured warping stress at the support for the two-channel section (Fig. 2(*a*)) varied from $\sigma$ = 145 to 690 lb per sq in.; for the four-plate section

FIG. 7

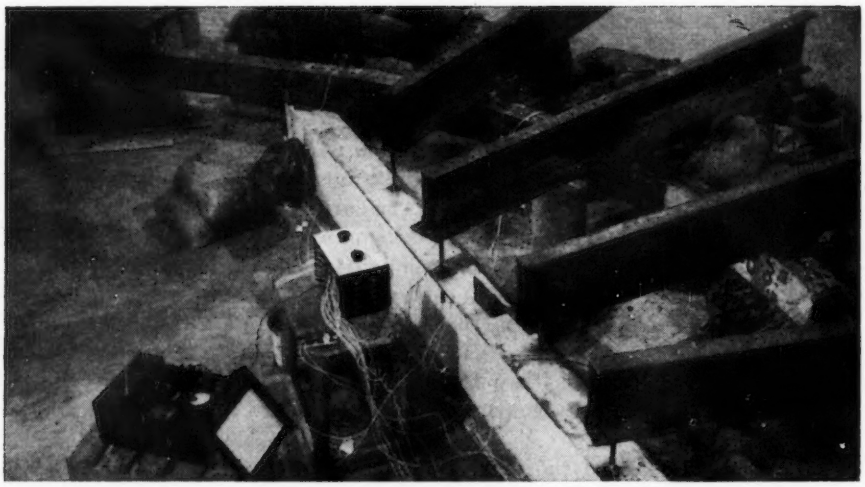

FIG. 8

(Fig. 2(*b*)) from $\sigma$ = 1628 to 2698 lb per sq in.; and for the I-section with side plates (Fig. 2(*c*)) from $\sigma$ = 710 to 1508 lb per sq in. The calculated values of this secondary effect (pounds per square inch) are 266, 620, and 220, respectively. The poor comparison between calculated and test results for rotation

and the secondary warping stress may be due to the intermittent welding of test members, slight resistance to torsional rotation at the supports, and inability of one section to retain its rectangular shape. In this latter respect it is noted that the largest warping stresses occurred in the four-plate section which offered the least resistance to change of cross section and the smallest

TABLE 1.—COMPARISON OF OBSERVED AND COMPUTED UNIT STRESSES (UNITS, POUNDS PER SQUARE INCH)

| Item | Description | Two Channels | | Four Plates | | I-Beam | |
|---|---|---|---|---|---|---|---|
| | | Measured | Computed | Measured | Computed | Measured | Computed |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 1 | Reference to Fig. 2 | Fig. 2(*a*) | | Fig. 2(*b*) | | Fig. 2(*c*) | |
| 2 | Loading on the beams, in pounds | 9,900 | | 13,510 | | 16,430 | |
| | Longitudinal stress due to bending: | | | | | | |
| 3 | End gages, 6 in. from support | 13,041 | 16,495 | 17,181 | 16,513 | 14,750 | 16,477 |
| 4 | Central gages, 6 in. from center of span | 8,124 | 10,443 | 9,599 | 10,455 | 11,192 | 10,432 |
| | Maximum shearing stress at outside face: | | | | | | |
| 5 | 19°21.69′ from the supports | 4,772 | 4,710 | 6,450 | 6,462 | 4,411 | 3,788 |

warping stresses occurred in the two-channel section which was welded at the center of the top and bottom.

Thus, by using this arrangement of continuing the beam over the supports, and by using a closed section, the warping stress is minimized to that of a secondary effect and the bending stresses at the support control the design. This condition is to be contrasted with the case of an I-beam curved horizontally through 120° and fixed at the supports, in which the stresses due to this effect are equal to, or larger than, the bending stresses (14).

## CONCLUSIONS

From the analysis of the experimental data taken on the testing of box sections, the following conclusions are offered for discussion:

1. Box sections are recommended for use in the design of beams subjected to torsion and bending, since they provide much greater torsional rigidity than the open section.

2. The design of a box section for a curved corner unsupported between column supports is recommended and is indicated in Appendix III.

3. The secondary effect (called "warping of the section") can be reduced by selecting dimensions such that the ratio of the thickness of the sides to their lengths are equal—that is, $t_1/b = t_2/h$.

4. Use of the section made up of two channels (Fig. 2(*a*)) is recommended over the section composed of four plates (Fig. 2(*b*)), since the two-channel section has greater resistance to change of cross-sectional shape.

5. The section consisting of an I-beam with side plates (Fig. 2(*c*)) may be reduced in cost by discontinuing the side plates over a part of the straight

section of beam on each end—the portion of beam being the length over which the required section modulus is smaller than that provided by the I-beam.

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### Appendix I.—Bibliography on Horizontally Curved Box Beams

(1) **"Zur Berechnung gekrümmter Träger,"** by W. L. Andree, *Der Eisenbau*, Vol. 9, 1918, pp. 184–190.

(2) **"Bethlehem Manual of Steel Construction,"** Bethlehem Steel Co., Bethlehem, Pa., 1934, pp. 279–289.

(3) **"Berechnung des senkrecht zu seiner Ebene Belastiten Bogenträgers,"** by Karl Federhofer, *Zeitschrift für Mathematik und Physik*, 1914, Vol. 62, pp. 40–63.

(4) **"Beitrag zur Berechnung gekrümmter Träger,"** by J. Hailer, *Die Bautechnik*, 1932, Vol. 10, p. 372.

(5) **"Der Kontinuierliche, halbkreisförmig gebogene und gleichmässig belastete Eisenbetonträger mit rechteckigem Querschnitt auf drei und vier gleich weit enfernten Stützen,"** by S. Hessler, *Beton und Eisen*, Vol. 29, 1930, pp. 149–154.

(6) **"Torsion of Flanged Members with Cross-Sections Restrained Against Warping,"** by H. M. Hill, *Technical Note No. 888*, National Advisory Committee for Aeronautics, March, 1943.

(7) **"Torsion of Rectangular Tubes,"** by William Hovgaard, *Journal of Applied Mechanics*, Vol. 4, No. 3, September, 1937, pp. 131–135.

(8) **"Zur Theorie Torsionfester Ringe,"** by B. G. Kannenberg, *Der Eisenbau*, Vol. 4, 1913, pp. 329–334.

(9) **"A Recurrence Formula for Shear-Lag Problems,"** by Paul Kuhn, *Technical Note No. 739*, National Advistory Committee for Aeronautics, December, 1939.

(10) **"Some Elementary Principles of Shell Stress Analysis with Notes on the Use of the Shear Center,"** by Paul Kuhn, *Technical Note No. 690*, National Advisory Committee for Aeronautics, March, 1939.

(11) **"Structural Beams in Torsion,"** by I. Lyse and B. G. Johnston, *Transactions*, ASCE, Vol. 101, 1936, p. 857. (a) Fig. 8, p. 865.

(12) **"Über Elastizität und Stabilität des geschlossen und offenen Kreisbogens,"** by R. Mayer, *Zeitschrift für Mathematik und Physik*, Vol. 61, 1913, pp. 302–308.

(13) **"Stresses in a Curved Beam Under Loads Normal to the Plane of Its Axis,"** by Robert B. B. Moorman, *Iowa State College Bulletin No. 145*, Iowa State College, Ames, Iowa, March, 1940.

(14) **"Stresses in a Uniformly Loaded Circular Arc I-Beam,"** by Robert B. B. Moorman, Vol. 48, No. 27, *Engineering Series No. 36*, University of Missouri, Columbia, Mo., 1947.

(15) **"The Stress Analysis of Bow Girders,"** by A. J. S. Pippard and F. L. Barrow, Department of Scientific and Industrial Research, *Building Research Board Technical Paper No. 1*, His Majesty's Stationery Office, London, 1926.

(16) **"Neure Probleme aus der Flugzeugstatik,"** by H. Reissner, Vol. 18, 1926, p. 384.

(17) **"The Torsion of Members Having Sections Common in Aircraft Construction,"** by George W. Trayer and H. W. March, *Technical Report No. 334*, National Advisory Committee for Aeronautics, March, 1930.

(18) **"Der Kreistrager,"** by G. Unold, *Forschungsarbeiten auf dem gebiete des Ingenieurwesens*, No. 255, Verein Deutscher Ingenieure, Berlin, Germany, 1922.

(19) **"Circles of Strain,"** by Joseph A. Wise, *Journal of the Aeronautical Sciences*, Vol. 7, No. 10, August, 1940, p. 438.

(20) **"Torsional Strength of Steel I-Sections,"** by C. R. Young and C. A. Hughes, *Bulletin 4*, School of Engineering Research, University of Toronto, Toronto, Ont., Canada, 1924, pp. 131–144.

## Appendix II.—Nomenclature

The following symbols, adopted for use in the paper and for the guidance of discussers, conform essentially with "American Standard Letter Symbols for Structural Analysis" (ASA-Z10.8–1942), prepared by a committee of American Standards Association with ASCE participation, and approved by the Association in 1942:

$A$ = a mathematical coefficient defined by Eq. 18.
$A_s$ = shear area, in square inches. ($b \times h$.)
$a$ = a substitution factor defined by Eq. 15*a*.
$b$ = width of section (center to center of side walls), in inches.
$B$ = a mathematical coefficient defined by Eq. 17.
$C$ = a substitution factor defined by Eq. 15*b*.
$D$ = a mathematical coefficient defined by Eq. 21.
$E$ = modulus of elasticity in pounds per square inch.
$E$ = a mathematical coefficient defined by Eq. 23.
$e$ = base of natural logarithms.
$f_b$ = allowable stress in bending.
$G$ = modulus of elasticity in shear, in pounds per square inch.
$h$ = depth of section (center to center of top and bottom thicknesses of the section), in inches.
$I$ = moment of inertia, in inches$^4$.
$J$ = torsional rigidity of section, in inches$^4$.
$l$ = length of beam from center of span to the support, in inches.
$M$ = resisting bending moment in beam, in inch-pounds.
$Q$ = statical moment of area about the neutral axis, in inches$^3$.

$q$ = shear flow, in pounds per inch.
$r$ = radius of curvature of the beam, in feet.
$S$ = section modulus, in inches$^3$.
$s$ = distance; $ds$ = small increment of length.
$T$ = external torque in inch-pounds; $T_{max}$ = maximum torque.
$t$ = thickness, in inches.
$t_1$ = top and bottom walls of sections.
$t_2$ = side walls of section.
$V$ = vertical shear, in pounds.
$w$ = uniform load, in pounds per foot of beam.
$x$, $y$, and $z$ = coordinate axes and corresponding distances.
$\alpha$ = angle measured in the horizontal plane.
$\delta$ = deflection, in inches.
$\sigma$ = unit stress due to bending, in pounds per square inch.
$\theta$ = angle measured in the horizontal plane.
$\tau$ = unit stress in shear, in pounds per square inch.
$\phi$ = angle of rotation, in radians.

## Appendix III.—Proposed Method of Design

*Statement of Problem.*—Design a horizontally curved beam as shown in Fig. 4 with a radius of curvature $r = 12$ ft, $w = 600$ lb per ft over the entire length from point C to point D, and $l_1 = 12$ ft. The maximum bending moments in the beam occur at the supports A and B. From Fig. 9($a$) and Eq. 2($b$): $M_A = M_B = -0.2146\, w\, r^2 = 222{,}497.28$ in.-lb. The maximum torques occur at $\theta = 19°12'$ from supports A and B on the curved length of the beam as shown in Fig. 9($c$) and by Eq. 2($i$): $T_{max} = 0.03312\, w\, r^2 = 34{,}338.82$ in.-lb. The vertical shear at point of maximum torque may be taken from the diagram in Fig. 9($b$), or computed, thus, for $\theta = 19°12'$: $V = 0.7854\, w\, r - \frac{19.2}{180}\pi\, w\, r = 2431.62$ lb.

The stress that approaches the allowable unit stress first is the unit stress created by bending at supports A and B. This stress will be increased by the warping of the section. Selection of the section will be based on the required section modulus as determined by $M_A$ and by keeping the term $\frac{1}{t_2 b} - \frac{1}{t_1 h}$ to a minimum. The required section modulus is $S = \frac{M}{f_b} = \frac{222{,}497.28}{20{,}000} = 11.1249$ in$^3$. (The 1946 specifications of the American Institute for Steel Construction, AISC, gives $f_b = 20{,}000$ lb per sq in.)

Try two channels, 6 in. by 2 in. by 13.0 lb, with dimensions as shown in Fig. 10, and other characteristics as follows: $S = 11.6$ in.$^3$; $I = 34.6$ in.$^4$; $b = 4.314 - 0.437 = 3.877$ in.; $h = 6.000 - 0.343 = 5.657$ in.; and $A_s = b\, h = 3.877 \times 5.657 = 21.9322$ sq in. The relations $\tau = \frac{T}{2\, A_s\, t_2}$ and $J = \frac{4\, A_s^2}{\oint \frac{ds}{t}}$ have been devel-

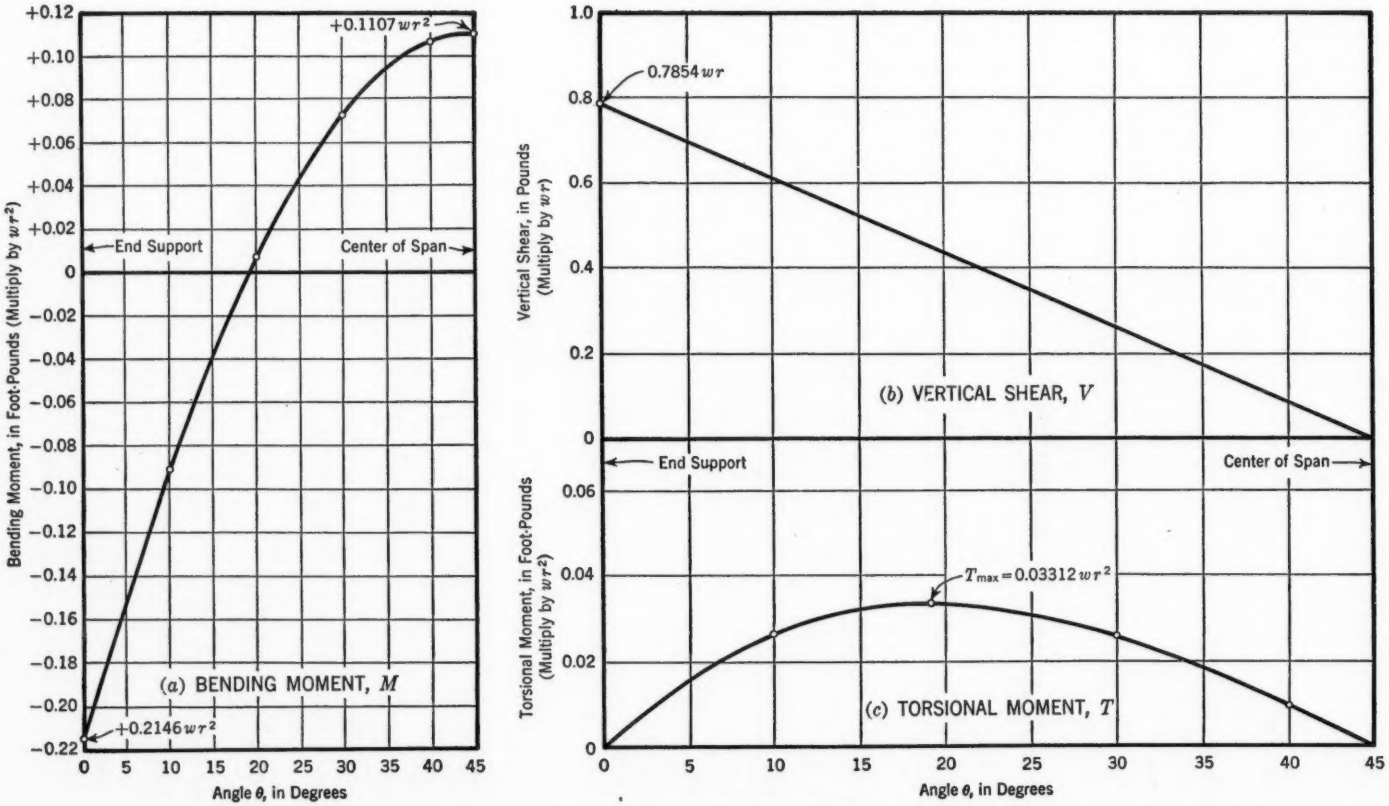


FIG. 9.—VARIATIONS IN MOMENT AND SHEAR OVER THE CURVED LENGTH OF THE BEAM, UNIFORMLY LOADED (LENGTH A B, FIG. 4)

oped by Paul Kuhn (10). According to Eq. 6, $J = \dfrac{4\,(21.9322)^2}{2\left(\dfrac{5.657}{0.437} + \dfrac{3.877}{0.343}\right)}$ $= 39.6746$ in$^4$.

The maximum shearing stress in the beam occurs where the torque is maximum ( $\theta = 19°12'$ ) and occurs on the outer side face of the curved beam. The maximum shearing stress (in pounds per square inch) is the sum of shearing stress due to torque and the shearing stress due to vertical shear:

$$\tau(\text{torque}) = \frac{T}{2\,A_s\,t_2} = \frac{34{,}338.82}{2 \times 21.9322 \times 0.437} = 1791.40 \text{ lb per sq in.}$$

$$\tau \text{ (vertical shear)} = \frac{V\,Q}{I\,t} = \frac{2431.62 \times 7.2704}{34.6 \times 2 \times 0.437} = 584.61 \text{ lb per sq in.}$$

in which

$$Q = 2 \times 3 \times 0.437 \times \frac{3}{2} + 3.44 \times 0.343\,(3 - 0.1715) = 7.2704 \text{ in}^3.$$

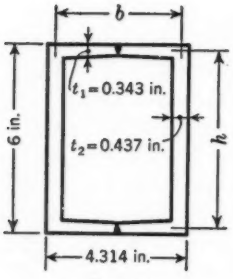

FIG. 10

The maximum shear stress is $\tau = 1791.40 + 584.61 = 2{,}376.01$ lb per sq in. The allowable shear stress (AISC specifications) is $\tau = 13{,}000$ lb per sq in.

The deflection of the beam at the center of the span equals the sum of three effects—(*a*) the change in slope at the support; (*b*) the deflection of the beam from the tangent to the elastic curve at the support due to moment; and (*c*) the deflection of the beam from the tangent to the elastic curve at the support due to torque.

The change in slope at the support, as calculated by the double integration method, is expressed as $\dfrac{1}{E\,I}\left(\dfrac{w\,l_1^3}{24} - 0.07153\,w\,r^2\,l_1\right)$. The slope at the support is $\dfrac{1}{29 \times 10^6 \times 34.6} \times \left(\dfrac{600 \times 12^3}{24} - 0.07153 \times 600 \times 12^3\right) 12^2 = -0.00444297$ radians; and $\delta$ (slope) = slope $\times\, r \times \sin 45° = -0.00444297 \times 12 \times 12 \times 0.70711 = -0.4524$ in.

The deflection of the beam from the tangent to the elastic curve at the support, due to moment (see Eq. 3*d*) is

$$\delta \text{ (moment)} - 0.015212\,\frac{w\,r^4}{E\,I} = -0.015212\,\frac{600 \times 12^4 \times 12^3}{29 \times 10^6 \times 34.6} = -0.32594 \text{ in.}$$

The deflection of the beam from the tangent to the elastic curve at the support due to torque (see Eq. 5*c*) is

$$\delta \text{ (torque)} = +0.0016845\,\frac{w\,r^4}{G\,J}$$

$$= +0.0016845\,\frac{600 \times 12^4 \times 12^3}{11.6 \times 10^6 \times 39.6746} = +0.07869 \text{ in.}$$

The total deflection at the center of the span is $-0.45240 - 0.32594 + 0.07869 = -0.69965$ in.

*Warping Effect of Beam.*—By Eq. 15*a*,

$$a^2 = \frac{19.2}{21.9322\,(6.4334 - 0.0855)} = 0.13790843$$

and $a = 0.3713602$. The term

$$\frac{1}{t_2 b} - \frac{1}{t_1 h} = \frac{1}{0.437 \times 3.877} - \frac{1}{0.343 \times 5.657} = 0.07486.$$

The maximum torque $T_{max} = 34{,}338.82$ in.-lb; and one half the length of the curved part of beam, $l$, is equal to $\frac{\pi}{4} r = \frac{\pi}{4} \times 12^2 = 113.0972$ in. Eq. 15*b* yields

$$C = -\frac{\frac{6\pi \times 34{,}338.816}{113.0972} \times 0.07486}{139.2228} = -3.0774.$$

By Eq. 17,

$$B = -\frac{C}{\frac{\pi^2}{l^2} + a^2} = \frac{3.0774}{0.000772 + 0.137908} = 22.1905$$

and, by Eq. 18, $A = -22.1905$. Substituting the values of these constants into Eq. 16 to determine warping $\sigma$ (support): $B = 22.1905$ lb per sq. in.

*Rotation of the Beam.*—Substituting the foregoing values in Eq. 21,

$$D = \frac{22.1905}{5.657 \times 10^6}\left(\frac{4}{41.7531} - \frac{1.43976}{23.2}\right) = 0.13236 \times 10^{-6}.$$

Making proper substitutions in Eq. 23,

$$E = 10^{-6}\,(+\,180.8642 + 2506.4437 + 0.6555 - 14.3976)$$
$$= 2{,}672.9938 \times 10^{-6} \text{ radians.}$$

Similarly, making appropriate substitutions in Eq. 24, for $z = 0$:

$$\phi = 10^{-6}\,(181.8762 + 2{,}504.1913 + 2{,}672.9938) = 5.35906 \times 10^{-3} \text{ radians}$$

and the rotation at the center due to torsion:

$$\phi = \frac{180}{\pi} \times 5.35906 \times 10^{-3} = 0.30705°.$$

The total rotation of the center of the beam is made up of the effects due to torsion, change in slope of support, and bending of the beam due to moment. The rotation due to change in slope is equal to slope $\times$ sin 45°; thus,

$$\phi \text{ (slope)} = 0.004443 \times 0.70711 \times \frac{180}{\pi} = 0.18000°.$$

The rotation due to the effect of bending the beam from the support to the center of span equals the expression for deflection due to moment with $r^3$ instead of $r^4$:

$$\phi \text{ (moment)} = 0.015212 \frac{w r^3}{E I} = 0.015212 \frac{600 \times 12^5}{29 \times 10^6 \times 34.6}$$

$$= 0.00226 \text{ radians} = 0.00226 \times \frac{180}{\pi} = 0.12969°.$$

The total rotation of the beam at the center of the span equals the sum of the three effects:

$$\phi \text{ (total)} = 0.30705° + 0.18000° + 0.12969° = 0.61674°.$$

*Welding of Channels.*—The two channels are connected by intermittent butt welds. The allowable unit stress for shear on throat area of butt welds (AISC Specifications) = 13,000 lb per sq in. The maximum shear flow $q = \frac{T_{max}}{2 A_s}$ = 782.84 lb per linear in. The capacity of a $\frac{3}{16}$-in. weld = $\frac{3}{16} \times 13{,}000 = 2438$ lb per linear inch of weld. Use an intermittent weld of 4 in. with 6-in. spaces between welds.